

Algebraic condition for decomposition of large-scale linear dynamic systems

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Abstract

The paper concerns the problem of decomposition of a large-scale linear dynamic system into two subsystems. An equivalent problem is to split the characteristic polynomial of the original system into two polynomials of lower degrees. Conditions are found concerning the coefficients of the original polynomial which must be fulfilled for its factorization. It is proved that knowledge of only one of the symmetric functions of those polynomials of lower degrees is sufficient for factorization of the characteristic polynomial of the original large-scale system. An algorithm for finding all the coefficients of the decomposed polynomials and a general condition of factorization are given. Examples of splitting the polynomials of the fifth and sixth degrees are discussed in detail.

Keywords: decomposition, algebraic condition, dynamic systems

1. Introduction

A dynamic system is a recent theoretical approach to the study of development. In its contemporary formulation, the theory grows directly from advances in understanding complex and nonlinear systems in physics and mathematics, but it also follows a long and rich tradition of systems thinking in biology and psychology. The term dynamic system, in its most generic form, means systems of elements that change over time. The more technical use, dynamical systems, refers to a class of mathematical equations that describe time-based systems with particular properties. These branches of mathematics include control theory, homotopy theory, and matroid theory. As a complement to these varied perspectives, the approach of this paper is the study of algebraic Conditions for decomposition of large-scale linear dynamic systems. There are many articles concerning decompositions of polynomials (Alagar and Thanh, 1985; Bartoni and Zippel, 1985; Borodin *et al.*, 1985; Coulter *et al.*, 2001; 1998; Gathen, 1990; Giesbrecht and May, 2005; Kozen *et al.*, [1, 2, 3, 4 & 5]1989; Kozen and Landau, 1989; Watt, 2008) [7, 8, 9, 10, 11, 12, 13, 15, 16, 21].

These methods deal with polynomials in numerical forms. The main idea is to find a decomposition $f(x) = g[h(x)]$. For example, let $f(x) = x^6 + 6x^4 + x^3 + 9x^2 + 3x - 5$. It is possible to decompose this polynomial, where $y = h(x) = x^3 + 3x$ and $g(x) = y^2 + y - 5$. For this type of decomposition, if possible, various algorithms are presented. In this article, a general method for decomposition of polynomials in the symbolic form $f(x) = \sum_{i=0}^n a_i s^i$, $a_i \in R^n$ is presented. Using computer algebra systems, it is possible to decompose polynomials of very high degrees.

Decomposition of large-scale linear systems into smaller dynamic systems is always possible if the eigenvalues of the state matrices which characterize the systems are known (Górecki and Popek, 1987) [14]. On the other hand, it is well known that, in general, algebraic equations of the degree $n \geq 5$ cannot be solved in radicals when their coefficients belong to the rational field; see the theorem of Ruffini and Abel.

1.1 Theorem 1. (Galois)

A necessary and sufficient condition for analytical solvability of an algebraic equation of the degree > 4 is that the all of its roots s_1, s_2, \dots, s_n could be represented in the rational field of its coefficients by a the rational function $f(s_1, s_2)$ of two of its roots s_1 and s_2 (Mostowski and Stark, 1954; Suszkiewicz, 1941; Perron, 1927) [4, 17, 20, 18].

Our problem is to find algebraic conditions which would enable us to decompose the characteristic polynomial of a large-scale dynamic system into two polynomials which altogether have the same roots as the original characteristic polynomial. Repeating such a decomposition (if possible), we can decompose the original system into more than two parts. It is shown that such a decomposition is possible if some additional relations are known between the coefficients or the roots of the factorial polynomials.

2. Problem Statement

Let us consider the characteristic polynomial

$$F(s) = s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n = 0, \quad (1)$$

Where s_i , $i = 1, 2, \dots, n$ are known real coefficients which may depend on some real parameters. Our goal is the following decomposition of the polynomial (1)

$$F(s) = G(s) H(s), \quad (2)$$

Where

$$G(s) = \sum_{k=0}^m b_k s^{m-k} \quad (3)$$

$$H(s) = \sum_{k=0}^{n-m} c_k s^{n-m-k} \quad (4)$$

$$a_0 = b_0 = c_0 = 1$$

3. Problem Solution

3.1 Decomposition of the fifth-degree algebraic equation into equations of the second and third degrees

Let us consider the characteristic polynomial

$$F(s) = s^5 + a_1 s^4 + a_2 s^3 + a_3 s^2 + a_4 s + a_5 = 0 \quad (5)$$

As for additional information, we assume that we know the sum of the two roots

$$s_1 + s_2 = A_1. \quad (6)$$

Then

$$s_3 + s_4 + s_5 = a_1 - A_1 = B_1 \quad (7)$$

Where B_1 is known, as calculated from (7)

Let

$$(s - s_1)(s - s_2) = s^2 - A_1 s + A_2 \quad (8)$$

$$(s - s_3)(s - s_4)(s - s_5) = s^3 - B_1 s^2 + B_2 s + B_3 \quad (9)$$

Knowing the coefficients a_1, \dots, a_5 and the sums A_1 and B_1 , we determine A_2 and B_2, B_3 .

We have

$$\begin{aligned} f(s) &= (s^2 - A_1 s + A_2)(s^3 - B_1 s^2 + B_2 s + B_3) \\ &= s^5 - (A_1 + B_1) s^4 + (A_2 + B_2 + A_1 B_1) s^3 + (B_3 - A_1 B_2 - A_2 B_1) s^2 + (A_2 B_2 - A_1 B_3) s + A_2 B_3 \end{aligned} \quad (10)$$

From (5) and (10), we obtain the following relations:

$$A_2 + B_2 = a_2 - A_1 B_1 \quad (11)$$

$$B_3 - A_1 B_2 - A_2 B_1 = a_3 \quad (12)$$

$$A_2 B_2 - A_1 B_3 = a_4 \quad (13)$$

$$A_2 B_3 = a_5 \quad (14)$$

We consider two cases:

Case 1

$A_1 \neq B_1$. From (11) and (12) we express A_2 and B_2 as linear functions of B_3, A_1 and B_1 , which are known:

$$A_2 = \frac{A_1(a_2 - A_1 B_1) + a_3 + B_3}{A_1 - B_1} \quad (15)$$

$$B_2 = \frac{-B_1(a_2 - A_1 B_1) + a_3 + B_3}{A_1 - B_1}. \quad (16)$$

Substituting (15) into (14), we obtain the equation of the second degree for calculation of B_3

$$B_3^2 + [A_1(a_2 - A_1 B_1) + a_3] B_3 - (A_1 - B_1) a_5 = 0. \quad (17)$$

After solving this equation, from (15) and (16) we calculate the coefficients A_2 and B_2 . In this way, decomposition of the

polynomial $f(s)$ into the factors of the second and third degrees is completed, and the solution of the equation of the fifth degree is reduced to solving equations of the second and third degrees. Substituting the coefficients A_2 and B_3 into (14), we obtain the relation between the coefficients a_1, \dots, a_5 which must be fulfilled as a consequence of the assumption (6) that

$$s_1 + s_2 = A_1$$

Case 2

$A_1 = B_1$. Equations (11) and (12) are

$$A_2 + B_2 = a_2 - A_1^2 \tag{18}$$

$$B_3 - A_1 (A_2 + B_2) = a_3 \tag{19}$$

From (18) and (19) we obtain

$$B_3 = a_3 + A_1(a_2 - A_1^2) \tag{20}$$

The coefficient B_3 is known.

In this case, from (14) we calculate the coefficient A_2 and from (18) the coefficients B_2 . All the coefficients are computed and the free equation (13) yields a relation between the coefficients $a_1 \dots a_5$. In this way, we decomposed the equation of the fifth degree into two equations which can be solved analytically.

3.2 Decomposition of the equation of the sixth degree into equations of the fourth and second degrees.

Let

$$f(s) = s^6 + a_1s^5 + a_2s^4 + a_3s^3 + a_4s^2 + a_5s + a_6 = 0. \tag{21}$$

We assume that

$$s_1 + s_2 = A_1, \tag{22}$$

Where A_1 is known. In this case, we have

$$s_3 + s_4 + s_5 + s_6 = a_1 - A_1 = B_1, \tag{23}$$

So that B_1 is calculated.

We can write

$$(s-s_1)(s-s_2) = s^2 - A_1s + A_2 \tag{24} \text{ and } (s - s_3)(s - s_4)(s - s_5)(s - s_6) = s^4 - B_1s^3 + B_2s^2 - B_3s + B_4. \tag{25}$$

Knowing the coefficients a_1, \dots, a_6 and the sums A_1 and B_1 , we determine A_2, B_2 and B_3, B_4 .

We have

$$f(s) = (s^2 - A_1s + A_2)(s^4 - B_1s^3 + B_2s^2 - B_3s + B_4) = s^6 + a_1s^5 + (A_2 + A_1B_1 + B_2)s^4 - (B_1A_2 + A_1B_2 + B_3)s^3 + (A_2B_2 + A_1B_3 + B_4)s^2 - (A_2B_3 + A_1B_4)s + A_2B_4 \tag{26}$$

From (21) and (26) we obtain

$$A_2 + A_1B_1 + B_2 = a_2 \tag{27}$$

$$B_1A_2 + A_1B_2 + B_3 = -a_3 \tag{28}$$

$$A_2B_2 + A_1B_3 + B_4 = a_4 \tag{29}$$

$$A_2B_3 + A_1B_4 = -a_5, \tag{30}$$

$$A_2B_4 = a_6 \tag{31}$$

It is necessary to consider the following two different cases.

Case 1:

$A_1 \neq B_1$ From (27) and (28) we express A_2 and B_2 as linear functions of B_3 :

$$A_2 = \frac{A_1(a_2 - A_1B_1) + a_3 + B_3}{A_1 - B_1} \tag{32}$$

$$B_2 = \frac{-B_1(a_2 - A_1B_1) + a_3 + B_3}{A_1 - B_1} \tag{33}$$

Substituting (32) and (33) into (29), we obtain B_4 as the following function of the second degree of the unknown B_3 :

$$B_4 = a_4 + \left[\frac{A_1(a_2 - A_1B_1) + a_3 + B_3}{A_1 - B_1} \right] \cdot \left[\frac{B_1(a_2 - A_1B_1) + a_3 + B_3}{A_1 - B_1} \right] - A_1B_3 \tag{34}$$

From (32) and (34) substituted into (30), we obtain the following equation of the second degree in the unknown B_3 :

$$\frac{A_1(a_2 - A_1B_1) + a_3 + B_3}{A_1 - B_1} B_3 + A_1 \left[a_4 - A_1B_3 + \frac{A_1(a_2 - A_1B_1) + a_3 + B_3}{A_1 - B_1} \frac{B_1(a_2 - A_1B_1) + a_3 + B_3}{A_1 - B_1} \right] = -a_5 \tag{35}$$

We can solve this equation and obtain B_3 . Then we compute A_2 and B_2 from (32) and (33), and, finally, B_4 from (34). In this way we decomposed $f(s)$ into factors of the second and fourth degrees. The problem is reduced to solving equations of the second and fourth degrees. Substituting the values of A_2 and B_4 into (31), we obtain the relation between the coefficients a_1, \dots, a_6 , which must be fulfilled as a consequence of the assumption (22) that $s_1 + s_2 = A_1$

Case 2:

$A_1 = B_1$. Equations (27) and (28) are as follows:

$$A_2 + B_2 = a_2 - A_1^2 \tag{36}$$

$$B_3 + A_1(A_2 + B_2) = -a_3. \tag{37}$$

From (36) and (37) we obtain

$$B_3 = -a_3 - A_1(a_2 - A_1 B_1), \tag{38}$$

And B_3 is known.

From (30) and (31) we compute A_2 and B_4 whereupon from (27) we determine the coefficient B_2 . All the unknown coefficients are calculated, and (29) yields the Desired relation between the coefficients a_1, \dots, a_6 .

3.3 Decomposition of the sixth-degree equation into two equations of the third degree.

We consider the relation (21) in the general form

$$f(s) = a_0 s^6 + a_1 s^5 + a_2 s^4 + a_3 s^3 + a_4 s^2 + a_5 s + a_6 = 0. \tag{39}$$

We assume that the following relation holds:

$$s_1 + s_2 + s_3 = s_4 + s_5 + s_6 \tag{40}$$

From this relation and (39), we conclude that

$$s_1 + s_2 + s_3 = s_4 + s_5 + s_6 = \frac{-a_1}{2a_0}. \tag{41}$$

We also have

$$\sum_{i \neq j} s_i s_j = (s_1 + s_2 + s_3)(s_4 + s_5 + s_6) + (s_1 s_2 + s_1 s_3 + s_2 s_3) + (s_4 s_5 + s_4 s_6 + s_5 s_6). \tag{42}$$

We obtain

$$s_1 s_2 + s_1 s_3 + s_2 s_3 = A_2 \tag{43}$$

$$s_4 s_5 + s_4 s_6 + s_5 s_6 = B_2 \tag{44}$$

$$s_1 s_2 s_3 = A_3 \tag{45}$$

$$s_4 s_5 s_6 = B_3 \tag{46}$$

Taking into account (41) – (44), we have

$$A_2 + B_2 + \frac{a_1^2}{4a_0^2} = \frac{a_2}{a_0}$$

This means that

$$A_2 + B_2 = \frac{a_2}{a_0} - \frac{a_1^2}{4a_0^2} \tag{47}$$

$$\text{and } \sum_{\substack{i \neq j \neq k \\ i \neq k}} s_i s_j s_k = (s_1 + s_2 + s_3)(s_4 s_5 + s_4 s_6 + s_5 s_6) + (s_4 + s_5 + s_6)(s_1 s_2 + s_1 s_3 + s_2 s_3) + s_1 s_2 s_3 + s_4 s_5 s_6,$$

Or

$$\frac{-a_1}{2a_0}(A_2 + B_2) + A_3 + B_3 = \frac{-a_3}{a_0}$$

From this and (47), we obtain

$$B_3 + A_3 = \frac{-a_3}{a_0} + \frac{a_1 a_2}{2a_0^2} - \frac{a_1^3}{8a_0^3} \tag{48}$$

Similarly,

$$\sum_{\substack{i \neq j \neq k \\ i \neq k \neq l \neq j}} s_i s_j s_k s_l = (s_1 s_2 + s_1 s_3 + s_2 s_3) (s_4 s_5 + s_4 s_6 + s_5 s_6) + (s_1 + s_2 + s_3) s_4 s_5 s_6 + (s_4 + s_5 + s_6) s_1 s_2 s_3. \quad (49)$$

Taking account (43) and (44), from (49) we have

$$A_2 B_2 - \frac{a_1}{2a_0} (A_3 + B_3) = \frac{a_4}{a_0} \quad (50)$$

Substitution of (48) into (50) gives

$$A_2 B_2 = \frac{a_4}{a_0} - \frac{a_1 a_3}{2a_0^2} + \frac{a_1^2 a_2}{4a_0^3} - \frac{a_1^4}{16a_0^4} \quad (51)$$

Now, from (47) and (51), we can determine A_2 and B_2
 Apart from that,

$$(s_1 s_2 s_3)(s_4 s_5 s_6) = \frac{a_6}{a_0} \quad (52)$$

This means that

$$A_3 B_3 = \frac{a_6}{a_0} \quad (53)$$

From (48) and (53), we can compute A_3 and B_3 .
 The roots s_1, s_2, s_3 are obtained as the solutions of the equation

$$s^3 + \frac{a_1}{2a_0} s^2 + A_2 s - A_3 = 0, \quad (54)$$

and the roots s_4, s_5, s_6 result from the equation

$$s^3 + \frac{a_1}{2a_0} s^2 + B_2 s - B_3 = 0. \quad (55)$$

Finally, we also have the equality

$$\sum_{i,j,k,l,m} s_i s_j s_k s_l s_m = s_1 s_2 s_3 (s_4 s_5 + s_4 s_6 + s_5 s_6) + s_4 s_5 s_6 (s_1 s_2 + s_1 s_3 + s_2 s_3). \quad (56)$$

From (56) and (43) – (46) we obtain

$$A_2 B_3 + B_2 A_3 = \frac{-a_5}{a_0} \quad (57)$$

Substituting the computed coefficients A_2, B_2, A_3 and B_3 into (57), we obtain the relation between the coefficients a_0, \dots, a_6 which results from the assumption.(40)

4. Main Result

4.1 Decomposition of the equation of the seventh degree into equations of the fifth and second degrees.

Let us consider the characteristic polynomial

$$F(s) = s^7 + a_1s^6 + a_2s^5 + a_3s^4 + a_4s^3 + a_5s^2 + a_6s + a_7 = 0 \tag{58}$$

Where

$a_1, a_2, a_3, a_4, a_5, a_6, a_7$ are coefficients of $s^6, s^5, s^4, s^3, s^2, s$ respectively?

As for additional information, we assume that we know the sum of the two roots

$$s_1 + s_2 = A_1. \tag{59}$$

Then

$$s_3 + s_4 + s_5 + s_6 + s_7 = a_1 - A_1 = B_1 \tag{60}$$

Where B_1 is known, as calculated from (60)

Let

$$(s - s_1)(s - s_2) = s^2 - A_1s + A_2 \tag{61}$$

$$(s - s_3)(s - s_4)(s - s_5)(s - s_6)(s - s_7) = s^5 - B_1s^4 + B_2s^3 + B_3s^2 + B_4s + B_5 \tag{62}$$

Therefore from (61) and (62). We decomposed the equation of the seventh degree into two equations which can be solved analytically. As in the above section of preliminaries we found that the equation (62) of fifth degree can be decomposed into equations of the second and third degrees.

4.2 Decomposition of the equation of the Eighth degree into equations of the Sixth and second degrees.

Let us consider the characteristic polynomial

$$F(s) = s^8 + a_1s^7 + a_2s^6 + a_3s^5 + a_4s^4 + a_5s^3 + a_6s^2 + a_7s + a_8 = 0 \tag{63}$$

Where

$a_1, a_2, a_3, a_4, a_5, a_6, a_7$ and a_8 are coefficients of $s^7, s^6, s^5, s^4, s^3, s^2, s$ respectively.

As for additional information, we assume that we know the sum of the two roots

$$s_1 + s_2 = A_1. \tag{64}$$

Then

$$s_3 + s_4 + s_5 + s_6 + s_7 + s_8 = a_1 - A_1 = B_1 \tag{65}$$

Where B_1 is known, as calculated from (65)

Let

$$(s - s_1)(s - s_2) = s^2 - A_1s + A_2 \tag{66}$$

$$(s - s_3)(s - s_4)(s - s_5)(s - s_6)(s - s_7)(s - s_8) = s^6 - B_1s^5 + B_2s^4 + B_3s^3 + B_4s^2 + B_5s + B_6 \tag{67}$$

Therefore, from (66) and (67). We decomposed the equation of the eighth degree into two equations which can be solved analytically. As in the above section of preliminaries we found that the equation (67) of sixth degree can be decomposed into equations of the second and fourth degrees.

5. Conclusion

The question of how often such decomposition is possible naturally arises. Notice that the product of the degrees of the components is simply the degree of the original polynomial. In particular, if the original polynomial is of prime degree, there is no nontrivial decomposition. Let us consider the simplest non-trivial case, i.e. degree Seven.

$$F(s) = s^7 + a_1s^6 + a_2s^5 + a_3s^4 + a_4s^3 + a_5s^2 + a_6s + a_7 = 0,$$

Where;

$a_1, a_2, a_3, a_4, a_5, a_6, a_7$ are coefficients of $s^6, s^5, s^4, s^3, s^2, s$ respectively, which has a solution as mentioned in the main result section.

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